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Algebraic structure and kernel of the Schrödinger equation

Wang Qinmou

Department of Physics, Anhui Normal University, Wuhu, Anhui, People's Republic of China

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Abstract. The algebraic structure of the Schrödinger equation is discussed and we take advantage of Lie algebra to derive the kernels of the Schrödinger equation.

There has been a revival of interest recently in the application of Lie's method in order to study the linear or non-linear evolution equation. In this paper we shall investigate the algebraic structure of the Schrödinger equation and take advantage of Lie algebra to derive the kernels of the equation.

We begin from the Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(x,t)=H\psi(x,t). \tag{1}$$

One can define the unitary operator

$$U(t_1,t_0)=\exp[-\mathrm{i}(t-t_0)H/\hbar] \tag{2}$$

and the formal solution of the equation is

$$\psi(x,t)=\exp(-\mathrm{i}\hbar H/\hbar)\psi(x,0). \tag{3}$$

We consider some special cases.

(i) *Free particle.* For a free particle $H=-(\hbar^2/2m)\partial_{xx}$ and using the formula (Suzuki 1983)

$$\mathrm{e}^{\alpha\partial_{xx}}f(x)=(4\pi\alpha)^{-1/2}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-y)^2}{4\alpha}\right)f(y)\,\mathrm{d}y \tag{4}$$

we arrive at

$$\psi(x,t)=\left(\frac{m}{2\pi\mathrm{i}\hbar t}\right)^{1/2}\int_{-\infty}^{\infty}\exp\left(-\frac{m(x-y)^2}{2\mathrm{i}\hbar t}\right)\psi(y,0)\,\mathrm{d}y. \tag{5}$$

We know that the kernel satisfies the relation

$$\psi(x_b,t_b)=\int_{-\infty}^{\infty}K(x_b,t_b;x_a,t_a)\psi(x_a,t_a)\,\mathrm{d}x_a \tag{6}$$

and so comparing (5) and (6) gives

$$K(x,t;y,0)=\left(\frac{m}{2\pi\mathrm{i}\hbar t}\right)^{1/2}\exp\left(-\frac{m(x-y)^2}{2\mathrm{i}\hbar t}\right). \tag{7}$$

(ii) *Harmonic oscillator.* For a harmonic oscillator $H=-(\hbar^2/2m)\partial_{xx}+\frac{1}{2}m\omega^2x^2$. Define

$$L_-=-\tfrac{1}{2}\partial_{xx} \qquad L_+=\tfrac{1}{2}x^2 \qquad L_3=\tfrac{1}{2}x\partial_x+\tfrac{1}{4}. \tag{8}$$

It is easily verified that these operators satisfy the commutation relation of the Lie algebra SU(1, 1), namely

$$[L_+, L_-]=2L_3 \qquad [L_3, L_\pm]=\pm L_\pm. \tag{9}$$

We can obtain the Baker-Campbell-Hausdorff relations of the group SU(1, 1) (Truax 1985, Fisher *et al* 1984):

$$\begin{aligned}\exp\left(-\frac{\mathrm{i}t}{\hbar}\left(-\frac{\hbar^2}{2m}\partial_{xx}+\tfrac{1}{2}m\omega^2x^2\right)\right)\\ &=\exp\left(-\mathrm{i}\frac{m\omega}{\hbar}\tan(\omega t)\tfrac{1}{2}x^2\right)\exp\left(-\frac{\mathrm{i}\hbar}{2m\omega}\sin(2\omega t)(-\tfrac{1}{2}\partial_{xx})\right)\\ &\quad\times\exp[-2\ln(\cos\omega t)(\tfrac{1}{2}x\partial_x+\tfrac{1}{4})].\end{aligned}\tag{10}$$

Employing (4) and

$$e^{\alpha x\partial_x}f(x)=f(\mathrm{e}^{\alpha}x) \tag{11}$$

gives us

$$\psi(x,t)=\left(\frac{m\omega}{2\mathrm{i}\pi\hbar\sin\omega t}\right)^{1/2}\int_{-\infty}^{\infty}\exp\left(\frac{\mathrm{i}m\omega}{2\hbar\sin\omega t}[(x^2+y^2)\cos\omega t-2xy]\right)\psi(y,0)\,\mathrm{d}y \tag{12}$$

and so

$$K(x,t;y,0)=\left(\frac{m}{2\pi\mathrm{i}\hbar\sin\omega t}\right)^{1/2}\exp\left(\frac{\mathrm{i}m\omega}{2\hbar\sin\omega t}[(x^2+y^2)\cos\omega t-2xy]\right). \tag{13}$$

(iii) *Harmonic oscillator and gravitational field.* For this problem $H=-(\hbar^2/2m)\partial_{xx}+\frac{1}{2}m\omega^2x^2+mgx$ or

$$H=-\frac{\hbar^2}{2m}\partial_{xx}+\tfrac{1}{2}m\omega^2(x+g/\omega^2)^2-\frac{mg^2}{2\omega^2}. \tag{14}$$

Let $x'=x+g/\omega^2$ and note that $[\partial_x, x']=[\partial_x, x]=1$.

We easily arrive at

$$\begin{aligned}\exp\left[-\frac{\mathrm{i}}{\hbar}t\left(-\frac{\hbar^2}{2m}\partial_{xx}+\tfrac{1}{2}m\omega^2x^2+mgx\right)\right]\\ &=\exp\left(\frac{\mathrm{i}}{\hbar}t\frac{mg^2}{2\omega^2}\right)\exp\left(-\frac{\mathrm{i}m\omega}{2\hbar}\tan(\omega t)(x+g/\omega^2)^2\right)\\ &\quad\times\exp\left(-\frac{\mathrm{i}\hbar}{2m\omega}\sin(2\omega t)(-\tfrac{1}{2}\partial_{xx})\right)\\ &\quad\times\exp\{-\ln\cos(\omega t)[(x+g/\omega^2)\partial_x+\tfrac{1}{2}]\}.\end{aligned}\tag{15}$$

Thus

$$\psi(x,t)=\left(\frac{m\omega}{2\pi i\hbar\sin\omega t}\right)^{1/2}\int_{-\infty}^{\infty}\exp\left(\frac{im\omega}{2\hbar\sin\omega t}[(x^2+y^2)\cos\omega t-2xy]\right.$$
$$-\frac{img}{\hbar\omega}+\tan(\tfrac{1}{2}\omega t)(x+y)$$
$$\left.+\frac{img}{\hbar\omega^2}[\tfrac{1}{2}t-(1/\omega)\tan(\tfrac{1}{2}\omega t)]\right)\psi(y,0)\,\mathrm{d}y \tag{16}$$

giving finally

$$K(x,t;y,0)=\left(\frac{m\omega}{2\pi i\hbar\sin\omega t}\right)^{1/2}\exp\left(\frac{im\omega}{2\hbar\sin\omega t}[(x^2+y^2)\cos\omega t-2xy]\right.$$
$$\left.-\frac{img}{\hbar\omega}\tan(\tfrac{1}{2}\omega t)(x+y)+\frac{img^2}{\hbar\omega^2}[\tfrac{1}{2}t-(1/\omega)\tan(\tfrac{1}{2}\omega t)]\right). \tag{17}$$

(iv) *Uniform field.* Here we have

$$H=-\frac{\hbar^2}{2m}\partial_{xx}+ex. \tag{18}$$

We notice that the commutation relations $[\partial_x^2, x]=2\,\partial_x$ and $[a^2, a^+]=2a$ are the same. Here a^+ and a are the creation and annihilation operators for the harmonic oscillator. From the formula (Katriel 1983)

$$\exp(\alpha a^+ +\beta a^r)=\mathrm{e}^{\alpha a^+}\exp\left(\sum_{i=1}^{r}\beta\alpha^i\binom{r}{i}\frac{1}{1+i}a^{r-i}\right) \tag{19}$$

we have

$$\exp\left[-\frac{i}{\hbar}t\left(-\frac{\hbar}{2m}\partial_{xx}+ex\right)\right]=\exp\left(-\frac{ietx}{\hbar}\right)\exp\left(\frac{i\hbar t}{2m}\partial_{xx}+\frac{et^2}{2m}\partial_x-\frac{it^3e^3}{6m\hbar}\right). \tag{20}$$

Using the well known relation

$$\mathrm{e}^{\alpha\partial_x}f(x)=f(x+\alpha) \tag{21}$$

we obtain

$$\psi(x,t)=\left(\frac{m}{2\pi i\hbar t}\right)^{1/2}\int_{-\infty}^{\infty}\exp\left[\frac{i}{\hbar}\left(\frac{m(x-y)^2}{2t}-\tfrac{1}{2}et(x+y)-\frac{e^2t^3}{24m}\right)\right]\psi(y,0)\,\mathrm{d}y. \tag{22}$$

Thus

$$K(x,t;y,0)=\left(\frac{m}{2\pi i\hbar t}\right)^{1/2}\exp\left[\frac{i}{\hbar}\left(\frac{m(x-y)^2}{2t}-\tfrac{1}{2}et(x+y)-\frac{e^2t^3}{24m}\right)\right]. \tag{23}$$

These results are the same as those obtained using Feynman's path integral method. Using the result of example (iii) and taking the limit as $\omega\to 0$ or $g\to 0$, one can obtain (23) or (13) from (17). This shows that the results obtained are the same as those derived by a variety of methods in this paper. If we compare (3) and (6), we clearly find that the path integral kernel is an integral transformation of the operator $U(t,0)$.

Thus it is important to develop the investigation of the algebraic structure of the differential equation.

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